Some mathematical problems in the theory of the stability of parallel flows

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By applying the method of initial values to the theory of stability of shear flows, Case has recently found certain results which are in apparent conflict with those obtained by the theory of normal modes. It is shown how these differences may be reconciled. Some new features in the theory of normal modes are also brought out. The relative merits of the two theories are compared.

1. Introduction

The usual theory of hydrodynamical stability is based on a study of the normal modes. There has been frequent controversy over this theory, because the behaviour of these normal modes are indeed very complicated at very large Reynolds numbers. For example, this writer (Lin 1955) has shown, from purely mathematical analysis, that '...there are certain damped solutions in a viscous fluid which, in the limit of vanishing viscosity, do not reduce to solutions of the inviscid equation throughout the whole region of the flow.' To those familiar with the theory of turbulence, such a behaviour is not at all surprising. Indeed, it is welcome as a concrete example of one of the fundamental characteristics of turbulent flow—i.e. that the flow field has an *intermittent spotty structure*. However, it is clear that the mathematical analysis leading to such a conclusion must be very complicated. In trying to clarify this situation as much as possible (and for other reasons), several attempts have been made to develop the theory of uniformly valid asymptotic solutions (see Langer 1940). Indeed, the old conclusions are found to be justified.

Recently, Case (1960, 1961) investigated the stability of parallel flows with respect to infinitesimal disturbances by considering the initial value problem. It is, of course, to be expected that there should be complete equivalence (cf. § 2) between this approach and the theory based on normal modes. Yet Case found that there were apparent inconsistencies, especially in connexion with the above-mentioned problem of the asymptotic limit as the Reynolds number becomes infinite. These apparent discrepancies disappear upon closer inspection. However, since they concern the fundamentals of the theories of stability and turbulence, and the problem is of general interest in the theory of singular perturbations of boundary-value problems of partial differential equations, it seems worth while to discuss the relationship between the two approaches in some detail so as to gain a deeper understanding of the issues. This is the purpose of the present note. The relative merits of the two approaches will be discussed in the concluding section of this paper.

It might be noted in passing, since the issue has been raised by Case (1961), that there is as yet no experimental evidence to suggest that the Navier–Stokes equations are inadequate to describe the phenomena of turbulent motion under ordinary conditions. For rarefied gases, the molecular structure should be considered once the Kolmogoroff scale of turbulence becomes comparable with the scales of molecular phenomena, e.g. the mean free path.

2. Some fundamental issues

We shall first formulate some of the fundamental questions that may be raised about the normal-mode theory of hydrodynamic stability, as a result of the conclusions reached by Case from the initial value theory. We consider the stability of a parallel flow†, for example, the pressure flow in a channel between parallel plates placed at $y = \pm 1$. The flow is in the x-direction with a velocity distribution $U(y) = 1-y^2$ (or some other parabolic function). The equation for small disturbances is

$$\frac{\partial \zeta'}{\partial t}+U\frac{\partial \zeta'}{\partial x}+v'\frac{d^2U}{\partial y^2} = \nu\Delta\zeta', \tag{2.1}$$

where ζ' is the disturbance vorticity, related to the disturbance stream function $\psi'(x,y,t)$ by

$$\zeta' = \frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\psi'}{\partial y^2} = \Delta\psi', \tag{2.2}$$

and the velocity components are given by

$$u' = \frac{\partial\psi'}{\partial y}, \quad v' = -\frac{\partial\psi'}{\partial x}. \tag{2.3}$$

The constant ν is the kinematic viscosity coefficient, or the inverse of the Reynolds number R in the present dimensionless formulation. The boundary conditions are

$$u' = v' = 0 \quad \text{at} \quad y = \pm 1. \tag{2.4}$$

The solution of (2.1) can be treated either in terms of the theory of normal modes or by the initial value approach. In the first approach, we superpose particular solutions of the form

$$\psi'(x,y,t) = \mathrm{Re}\{\phi(y)\,e^{i\alpha(x-ct)}\}, \tag{2.5}$$

where $\phi(y)$ satisfies the familiar Orr–Sommerfeld equation

$$\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi = i(\alpha/\nu)\,[(U-c)\,(\phi''-\alpha^2\phi)-U''\phi], \tag{2.6}$$

which is to be solved with the boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = 0. \tag{2.7}$$

In the initial value approach used by Case, we consider the Laplace transform of $\psi'(x,y,t)$ with respect to t and its Fourier transform with respect to x (α real, $\mathrm{Re}(p) > 0$):

$$\overline{\Psi}'\,(y;\alpha,p) = \iint \psi'(x,y,t)\,e^{-i\alpha x}dx\,e^{-pt}dt. \tag{2.8}$$

† Since Case did not formulate his theory in terms of the stream function, we develop below such a formulation in some detail to facilitate comparison with the normal-mode theory.

After solving for $\overline{\psi}'(y;\alpha,p)$ in terms of the initial conditions and the boundary conditions, we calculate $\psi'(x,y,t)$ by the inverse transform

$$\psi'(x,y,t) = \iint \overline{\psi}'(y,\alpha,p)\, e^{-i\alpha x} d\alpha\, e^{pt} dp, \tag{2.9}$$

where the integration with respect to p is taken along the line $\mathrm{Re}(p) = p_0 > 0$ in the direction of increasing $\mathrm{Im}(p)$. The normal mode representation is obtained by evaluating the p-integral at the singularities of $\overline{\psi}'$ in the p-plane by the theory of residues.

Apparent differences arise when one considers the *inviscid problem*, which is obtained by *formally* putting $\nu = 0$ in the above formulation and dropping the boundary condition $u' = 0$ in (2.4). For this case, Case has shown, by the consideration of the initial value problem, that normal modes associated with certain *continuous* spectra† of eigenvalues must be included. The eigenfunctions found are continuous functions with discontinuous first derivatives.

Case has further shown that for a set of initial conditions *independent of the Reynolds' number*, the solution of the viscous problem, in the limit $\nu \to 0$, approaches the solution of the inviscid problem except for a boundary-layer correction near the walls.

It might be tempting to suggest, as was done at one time, that the above conclusions combined would contradict the existence of the type of solutions described by this writer, as quoted in § 1. However, since the type of solutions considered, being of the type $\mathrm{Re}\{\phi(y,\nu)\, e^{i\alpha(x-ct)}\}$, is *dependent* on ν at *all* times, it is clear that they fall outside the scope of Case's theorem, because of his restriction on the initial conditions. There is therefore no contradiction. (In this connexion, it should be emphasized that the existence of the type of solution under discussion was established on *mathematical* grounds and *supplemented* by *physical* arguments—not the reverse.)

However, other questions may be raised about the theory of normal modes. For example, since the inviscid problem has normal modes associated with a continuous spectrum, and the solution of the viscous problem in general approaches the inviscid solutions as $\nu \to 0$, one may naturally pose the following question:

(1) are there continuous spectra of eigenvalues for the Orr–Sommerfeld equation?

We shall show in § 4 that the answer is negative. This answer leads to the next question:

(2) what is the viscous solution that corresponds to an inviscid normal mode, especially the one associated with the continuous spectrum?

We may also pose this question in terms of an initial value problem. Suppose we start with a set of *initial conditions* corresponding to an inviscid normal mode $\psi_0'(x,y,t) = \mathrm{Re}\{\phi_0(y)\, e^{i\alpha(x-ct)}\}$, i.e. suppose we require

$$\psi'(x,y,0) = \phi_{0r}(y)\cos\alpha x - \phi_{0i}(y)\sin\alpha x,$$

† The importance of studying the continuous spectra was pointed out earlier by Friedrichs (1941), but he did not give any detailed analysis.

and we consider the solution of (2.1) both in the inviscid case and in the viscous case. In the inviscid case, we naturally get the solution $\psi_0'(x,y,t)$ back. In the viscous case, the solution should also be close to $\psi_0'(x,y,t)$, according to Case. If there exists a normal mode

$$\psi_N'(x,y,t) = \mathrm{Re}\{\phi_N(y,\nu)\, e^{i\alpha(x-ct)}\}$$

that approaches the normal mode $\psi_0'(x,y,t)$, the natural answer to the initial value problem is then $\psi'(x,y,t) = \psi_N'(x,y,t) +$ small correction terms. But suppose such a normal mode does not exist,† what is the nature of the solution? The logical conjecture is also at hand; namely, the solutions should be represented as an infinite sum of normal modes (as any solution can be so represented), the sum being *not* dominated by a single mode. But is this logical answer also *plausible*?

In the following section, we shall show that this answer is indeed plausible by considering some simple examples where the solutions are explicitly known. These examples will also show the various ways in which the eigenfunctions of the viscous problem can behave in the limit of vanishing viscosity. In particular, we shall see that the limiting eigenfunctions may still form a discrete set while the inviscid problem has a continuous set of eigenfunctions. It is also possible for the inviscid problem to have eigenfunctions, while the limit of every viscous eigenfunction does not exist.

3. Examples of singular perturbation

In this section we shall examine the limiting behaviour of the solution of certain well-known parabolic equations as the 'diffusion coefficient' approaches zero. These properties are essentially well known; here they are presented in such a manner as to bring out the points of greatest interest to the issues at hand.

Example I. Consider the diffusion equation (which is applicable to the unsteady parallel flow between two parallel plates)

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{3.1}$$

with the boundary conditions $u(0) = u(1) = 0$. The 'inviscid' problem is simply

$$\frac{\partial u_0}{\partial t} = 0 \tag{3.2}$$

with no boundary conditions required. The normal modes (solutions obtained by the method of separation of variables) in the viscous case are

$$u_n(y,t,\nu) = \sin n\pi y\, e^{-n^2\pi^2\nu t}, \tag{3.3}$$

where n must take on the *discrete* set of values $1, 2, 3, \ldots$. For the inviscid equation (3.2), any continuous function $f(y)$ is a normal mode.

† This occurs not only in the case of continuous spectra of eigenvalues but also in the case when the inviscid problem has an unstable mode. The complex conjugate operation yields a damped inviscid solution

$$\tilde{\psi}'(x,y,t) = \mathrm{Re}\,\{\phi_0^*(y)\, e^{i(x-c^*t)}\}$$

which has no counterpart among the normal modes of this viscous problem.

In the limit $\nu \to 0$, the normal modes (3.3) become

$$u_n(y, t, 0) = \sin n\pi y \quad (n = 1, 2, 3, \ldots). \tag{3.4}$$

This is but one *discrete* set of normal modes of the *continuous* set $f(y)$ of the normal modes of (3.2). *Thus, the existence of a continuous set of eigenvalues and eigenfunctions in the inviscid case does not necessarily imply the existence of a corresponding set in the viscous case, even in the inviscid limit.*

Consider now the initial value problem, with

$$u(y, 0) = F(y). \tag{3.5}$$

Clearly, the inviscid solution is simply

$$u_0(y, t) = F(y). \tag{3.6}$$

The viscous solution is in general given by the series

$$u(y, t) = \sum_{n=1}^{\infty} A_n \sin n\pi y \, e^{-n^2\pi^2\nu t}, \tag{3.7}$$

where A_n is given by

$$A_n = \frac{2}{\pi}\int_0^{\pi} F(y) \sin n\pi y \, dy. \tag{3.8}$$

Clearly, as $\nu \to 0$, the viscous solution (3.7) approaches the inviscid solution (3.6) for all finite t. This is the gist of the theorem proved by Case for the hydrodynamical problem. On the other hand, if the initial condition (3.5) should contain ν, e.g. if the condition were $u(y, 0) = \sin(y/\nu)$, such a correspondence would not exist.

We note in passing that the limiting processes $\nu \to 0$ and $t \to \infty$ are not interchangeable for the solution (3.7), exactly as Case (1961) noted in his treatment of the hydrodynamical problem.

The solution (3.7) shows clearly that *all* the normal modes are in general excited in an initial value problem in the viscous case, even though the solution in the inviscid case is just a *single* mode. [The initial condition $u(y, 0) = \delta(y - y_c)$ yields the set of coefficients $A_n = \sin n\pi y_c$, which remain finite even as $n \to \infty$. In this case, a clear description of the solution for small t is the spread of a singularity, and not conveniently given by the method of normal modes. Our understanding can also be improved by examining the limiting process $\nu \to 0$ in the Laplace integral representation of the solution (see Appendix).]

Exceptional cases occur when the initial conditions correspond to the *inviscid limit* of a single viscous normal mode, $\sin n\pi y$ (or a finite sum of such terms). In this case, only that particular mode is excited. This example thus gives plausibility to the answers given at the end of the last section.

We note that the *inviscid limit* of the viscous normal modes *happens* to form a complete set of *inviscid* normal modes. In the next example, we shall see that *all* the viscous normal modes do not even have inviscid limits.

Example II. Consider now the equation

$$\frac{\partial u}{\partial t} + k\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{3.9}$$

The normal modes are

$$u_n(y) = w_n(y)\, e^{p_n t}, \tag{3.10}$$

with

$$w_n = e^{(k/2\nu)\, y} \sin n\pi y,$$

$$p_n = -\frac{k^2}{4\nu} - \nu n^2 \pi^2. \tag{3.11}$$

The inviscid problem has the normal modes

$$u = e^{\alpha y}\, e^{-k\alpha t}. \tag{3.12}$$

For the initial condition $e^{\alpha y}$, the viscous problem has a solution of the form

$$u(y,t,\nu) = \sum_1^\infty A_n(\nu)\, w_n(y,\nu)\, e^{p_n(\nu)t},$$

where the dependence of A_n, w_n and p_n on ν is specifically noted. It is obviously a complicated problem if one tries to calculate the limit of $u(y,t,\nu)$ as $\nu \to 0$ from the above series, since each *individual eigenfunction* $w_n(y,\nu)$ *does not have a limit as* $\nu \to 0$.

4. Continuous and discrete characteristic values

The examples examined in the last section serve to illustrate the type of behaviour that may occur in the asymptotic limit of the solution of a partial differential equation of the second order. Since the hydrodynamical problem involves an equation of the fourth order, it is not surprising that the behaviour of the solution could be even more complicated. (See §5 for the description of some important normal modes.) With this general background, let us return to the characteristic value problem (2.6), (2.7), and consider the question of a continuous spectrum of characteristic values.

Since the equation (2.6) is regular in the independent variable y as well as in the parameters R, α, c, there exists a fundamental system of four solutions which are analytic in y and the parameters R, α, c. Indeed they are entire functions. Furthermore since the boundary conditions (2.7) are to be satisfied for finite values of y, the secular equation for the determination of eigenvalues

$$F(R,\alpha,c) = 0 \tag{4.1}$$

is an entire function of the three parameters R, α, c, regarded as complex variables. If now there were a continuous set of eigenvalues for c, with R, α given, the equation (4.1) must reduce to an identity, by the theory of analytic continuation. Thus, the eigenvalues of the problem defined by (2.6) and (2.7) must all be *discrete*.

The above is merely an elementary proof, for our special case, of a general theorem for regular operators. The conclusion is thus obvious once it is pointed out.

Continuous spectra are expected for the inviscid equation,

$$(U-c)(\phi_0'' - \alpha^2\phi_0) - U''\phi_0 = 0, \tag{4.2}$$

(at least for the case $U'' = 2 \neq 0$), with the boundary conditions

$$\phi_0(\pm 1) = 0, \tag{4.3}$$

because the operator is now singular. Let us try to see this fact more directly in terms of the solutions of equation (4.2). This equation is well known to have solutions of the forms

$$\phi_1(y) = (y-y_c)[1+\ldots] \tag{4.4}$$

and

$$\phi_2(y) = 1+\ldots+\frac{U''_c}{U'_c}\phi_1(y)\log(y-y_c), \tag{4.5}$$

where $1+\ldots$ denotes a regular power series in $(y-y_c)$, and $U(y_c) = c$. In the case of real eigenvalues c, the *real* and *imaginary* parts of $\phi_2(y)$ are separately solutions of (4.2). The real part is the *continuous* function

$$\phi_{2r}(y) = 1+\ldots+\frac{U''_c}{U'_c}\phi_1(y)\log|y-y_c|, \tag{4.6}$$

and the imaginary part is another *continuous* function

$$\left.\begin{aligned}\phi_{2i}(y) &= 0, & y \geqslant y_c;\\ \phi_{2i}(y) &= -\pi\frac{U''_c}{U'_c}\phi_1(y) & y \leqslant y_c.\end{aligned}\right\} \tag{4.7}$$

We have thus *three* linearly independent solutions (4.4), (4.6) and (4.7) by permitting the solutions to have discontinuous or singular derivatives. With three linearly independent solutions for (4.2) and two boundary conditions (4.3), it is obvious that the characteristic values have a continuous spectrum.

However, if one regards the inviscid solution as the limit of the viscous solution as $\nu \to 0$, only the linear combination $\phi_{2r}(y)+i\phi_{2i}(y)$ is a solution, not $\phi_{2r}(y)$ and $\phi_{2i}(y)$ separately. Continuous spectra are therefore not obtained. The point is that the inviscid limit *does* exist in a properly defined region,† and remains an *analytic* function of the complex variable y. (This region, however, does not necessarily include the complete physical region.) Thus, the solution for $y > y_c$, when continued to the range $y < y_c$ must represent the same analytic function. The functions $\phi_r(y)$ and $\phi_i(y)$ do not satisfy this requirement.

5. Concluding remarks

Let us now summarize our conclusions and compare the relative merits of the methods of normal modes and initial values.

(i) Mathematically speaking, it should first be borne in mind that the Laplace integral representation of the solution can be decomposed into an infinite series of normal modes by the method of residues, and that the two approaches are essentially equivalent (see (vi) below).

(ii) Secondly, a great deal of apparent contradictions can be avoided if the following facts are kept in mind. A normal mode in the inviscid theory may not be the limit of a normal mode in the viscous theory. Conversely, a normal mode in the viscous theory may not have an inviscid limit. These conclusions are based on mathematical investigations of the solutions of the Orr–Sommerfeld equation. They are illustrated by simple examples in § 3. The calculations in the Appendix

† This region is defined by two of the three lines $\mathrm{Re}\int_{y_c}^{y}\sqrt{\{i(U-c)\}}\,dy = 0$, chosen in such a manner as to include the two end points of our boundary-value problem. See Lin (1955, figure 8·1, p. 130).

illustrate fully the lack of direct correspondence between the inviscid eigenfunctions and the inviscid limit of the viscous ones.

(iii) If we are dealing with disturbances where the viscous effects are restricted to a boundary-layer region of the order of $(\alpha R)^{-\frac{1}{2}}$ near the solid surface, the initial value approach leads to the familiar boundary-layer theory; i.e. the solution is composed of an inviscid solution plus a boundary-layer correction. In this method of description, the normal modes belonging to the continuous spectrum (or spectra), which occur in the inviscid case, must be used. This description is simpler than in terms of the normal modes in the viscous theory (which are discrete). The type of simplification is similar to that occurring with the examples of § 3.

(iv) However, there are important types of disturbances not of the type described above, In particular, the unstable disturbances responsible for the initiation of turbulence in the channel or the boundary layer are of a different type. These disturbances extend over a layer of the order of $(\alpha R)^{-\frac{1}{3}}$, which is thicker than the layer in the previous case. For these disturbances, the viscous forces have a well-known destabilizing influence typical of the instability of shear flows.

(v) There are also exponentially damped normal modes which have the intermittent spotty structure characteristic of turbulence. There are also normal modes which have no inviscid limit anywhere in the physical region. When, to attempt a description of the transition process, the non-linear theory is considered these normal modes are expected to play an important role.

(vi) The initial value theory has not yet been developed to study the two types of disturbances just mentioned. It would be interesting to carry out these investigations to see whether different features can be revealed by the Laplace integral representation. In particular, it should be possible to bring out the destabilizing influence of the viscous forces mentioned in (iv). It would appear, from the fact that there are only a finite number of unstable modes, that the two methods would yield essentially the same results with regard to the unstable behaviour of the flow. On the other hand, it is likely that one would find only stable disturbances for the flow through a channel, if the viscous forces are restricted to boundary-layer regions which have a thickness of the order of $(\alpha R)^{-\frac{1}{2}}$ (cf. Lin & Rabenstein 1960).

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Appendix A

To get a closer comparison with Case's investigation (1961) of the hydrodynamical problem, let us explicitly use the method of the Laplace transformation (for the equation (3.1)). The solution is then given by

$$u(y,t) = \frac{1}{2\pi i}\int_C \overline{u}(y,p)\, e^{pt} dp, \tag{A 1}$$

where $\bar{u}(y, p)$ satisfies the differential equation

$$\frac{d^2\bar{u}}{dy^2} - q^2\bar{u} = \frac{1}{k}u(y, 0), \quad q^2 = \frac{p}{\nu}, \tag{A 2}$$

and the contour C is the line $\mathrm{Re}(p) = p_0 > 0$, taken in the direction of increasing $\mathrm{Im}(p)$. For the purposes in hand, we need only consider the simple case $u(y, 0) = \text{const}$. We then have

$$\bar{u} = \frac{1}{p \sinh q}\{\sinh qy + \sinh q(1-y) - \sinh q\}. \tag{A 3}$$

We make two observations.

(i) If we deform the contour C so as to evaluate the integral (A 1) in terms of the residues at the poles of (A 3), $p = -\nu n^2\pi^2 (n = 1, 2, \ldots)$, we recover the series solution (3.7) in terms of the eigenfunctions. Notice that $\bar{u}$ does not have a singularity at $p = 0$.

(ii) In the limit $\nu \to 0$, the singularities of (A 3) become dense along the whole negative part of the real axis in the p-plane, but the limiting function is

$$\lim_{\nu \to 0} \bar{u} = -\frac{1}{p} \quad \text{for} \quad 0 < y < 1. \tag{A 4}$$

This limiting form (A 4) agrees with the inviscid transform

$$\bar{u}_0 = -\frac{1}{p}, \tag{A 5}$$

but its singularity lies at $p = 0$, and does not coincide with any one of the singularities of (A 3) for $\nu > 0$.

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